Laplace Transform

Definition

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

Notice that for Laplace transform, t is a real variable, and often it is time. s is in general complex.

The inverse Laplace transform is given by

$$f(t) = \lim_{k \to \infty} \frac{1}{2\pi i} \int_{\sigma - ik}^{\sigma + ik} F(s) e^{st} ds$$

where $|f(t)| \leq e^{Mt}$ for some positive real M, and $\sigma > M$. The integral requires contour integration, which consists of part of a circle and a vertical line (Bromwich contour), and the line part is what we want.
Notice the similarity between the above expression and the inverse Fourier transform. For inverse Fourier transform $\sigma = 0$ and $s \to i\omega$, so the exponential-order divergence is not allowed in Fourier transform.

The Laplace transform and inverse are unique, except for null functions (whose transform is 0). The uniqueness allows for interchanging between the two spaces t and s.

Properties

- $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$
- If there exists constants M and α such that $|f(t)| \leq Me^{\alpha t}$ for all t, then
$$\mathcal{L}[e^{-bt} f(t)] = F(s + b)$$
- $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$
- $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$
- $\mathcal{L}[f'(t)] = -f(0) + sF(s)$
- $\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$
- $\mathcal{L}[H(t - t_0) f(t - t_0)] = e^{-st_0} F(s)$, where H is the Heaviside's function.
- $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$
- $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

- $\mathcal{L}[\delta(t)] = 1$
- Suppose $f(t)$ has period T, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

Convolution

The definition is

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Some properties are

- $f * g = g * f$
- $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, this identity is very useful in finding the inverse Laplace transform.

ODE Applications

Since Laplace transform is linear, it can only be used to solve linear differential equations. The following example shows the standard routine.

Example Solve the differential equation

$$\frac{dx}{dt} + 3x = \cos(3t), \text{ with } x(0) = 0$$

Solution We take the Laplace transform to obtain

$$-x(0) + sX + 3X = \frac{s}{s^2 + 9}$$

With the initial condition, we have

$$X = \frac{s}{(s+3)(s^2+9)}$$

We see that this is for convolution. We get

$$x = \mathcal{L}^{-1}(X) = e^{-3t} * \cos(3t) = \int_0^t e^{-3(t-\tau)} \cos(3\tau)d\tau$$

The integral yields

$$x = \frac{1}{6}\left[\cos(3t) + \sin(3t) - e^{-3t}\right]$$

Laplace transform can also be used to solve PDEs, but it is of less importance.